NOETHERIAN QUOTIENTS OF THE ALGEBRA OF PARTIAL DIFFERENCE POLYNOMIALS AND GRÖBNER BASES OF SYMMETRIC IDEALS

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ABSTRACT. A Noetherianity criterion is provided in this paper for the quotients of the algebra of partial difference polynomials with respect to its difference ideals. After we introduce a Gröbner bases theory for ideals generated by difference polynomials, this implies the existence of finite such bases for the difference ideals containing elements with suitable linear leading monomials. As a by-product, one obtains computational methods for ideals of finitely generated polynomial rings that are invariant under the action of finite dimensional commutative algebras and in particular of finite abelian groups.

1. INTRODUCTION

The theory of difference algebras (see books [7, 15, 22] and references therein) was introduced in the 1930s by the mathematician Joseph Fels Ritt at the same time of the theory of differential algebras. Indeed, for a quite long time, difference algebras has attracted less interest among researchers in comparison with differential ones despite the fact that numerical integration of differential equations relies on solving finite difference equations. The rapid development of symbolic computation and computer algebra in the last decade of the previous century gave rise to rather intensive algorithmic research in differential algebras and to creation of sophisticated software as the *diffalg* library [3], implementing the Rosenfeld-Gröbner algorithm and included in MAPLE and the package LDA [13]. At the same time, except for algorithmization and implementation in MAPLE of the shift algebra of linear operators [6] as a part of the package ORE_ALGEBRA, practically nothing has been developed in computer algebra in relation to difference algebras. Nevertheless, in the last few years, the number of applications of the theory and the methods of difference algebras is increased fastly. For instance, it turned out that difference Gröbner bases may provide a very useful algorithmic tool for reduction of multi-loop Feynman integrals in high energy physics [9], for automatic generation of finite difference approximations to partial differential equations [11, 21] and for the consistency analysis of these approximations [10, 12]. Another source of applications is related to the notion of "letterplace correspondence" [16, 17, 19] which transforms non-commutative computations for presented groups and algebras into analogue computations for ordinary difference algebras. As a result of all this use, a number of computer algebra packages implementing involutive and Buchberger's algorithms

for computing difference Gröbner bases has been developed (see [10, 13, 18] and reference therein). A major drawback in these computations, as for the differential case, is that such bases may be infinite owing to Non-Noetherianity of the algebra of difference polynomials. In fact, if X is a finite set and Σ denotes a multiplicative monoid isomorphic to $(\mathbb{N}^r, +)$ then the algebra of difference polynomials is by definition the polynomial ring P in the infinite set of variables $X \times \Sigma$. Then, to provide the termination of the algorithms computing Gröbner bases in P at least in some significant cases, we propose in this paper essentially two solutions. One consists in defining an appropriate grading for P that allows finite truncated computations for difference ideals $J \subset P$ generated by a finite number of homogeneous elements. For monomial orderings of P that are compatible with such grading this implies a criterion, valid also for the non-graded case, able to certify the completeness of a finite Gröbner basis computed on a finite number of variables of P. After the algebra of partial difference polynomials and its Gröbner bases are introduced in Section 2 and 3, this approach is described in Section 4 and an illustrative example based on the approximation of the Navier-Stokes equations is given in Section 5. A second solution to the termination problem consists in requiring that the difference ideal J contains elements with suitable linear leading monomials which corresponds to have the Noetherian property for the quotient algebra P/J. Some similar ideas appeared for the differential case in [5, 24]. One finds this second approach in Section 6. It is interesting to note that an important class of such Noetherian quotient algebras is given by polynomial rings P' in a finite number of variables which are under the action of a tensor product of a finite number of finite dimensional algebras generated by single elements. These finite dimensional commutative algebras includes, for instance, group algebras of finite abelian groups and hence, as a by-product of the theory of difference Gröbner bases, one obtains a theory for Gröbner bases of ideals of P' that are invariant under the action of such groups or algebras (see also [15, 23]). These ideas are presented in Section 7 and a simple application is described in Section 8. Finally, in Section 9 one finds conclusions and hints for further developments of this research.

2. Algebras of difference polynomials

Let $(\Sigma, \cdot)$ be a monoid with identity element 1 and let K be a field. We denote by $\mathrm{End}(K)$ the monoid of ring endomomorphisms of K. We say that Σ *acts on* K or equivalently that K is a Σ-*field* if there exists a monoid homomorphism $\rho : \Sigma \to \mathrm{End}(K)$. In this case, we denote $\sigma \cdot \alpha = \rho(\sigma)(\alpha)$, for all $\sigma \in \Sigma$ and $\alpha \in K$. Starting from now, we assume that K is a Σ-field. Let A be a commutative K-algebra. We say that A is a Σ-*algebra* if there is a monoid homomorphism $\rho' : \Sigma \to \mathrm{End}(A)$ extending $\rho : \Sigma \to \mathrm{End}(K)$ that is $\rho'(\sigma)(\alpha) = \rho(\sigma)(\alpha)$ for all $\sigma \in \Sigma$ and $\alpha \in K$. Let $B \subset A$ be a K-subalgebra of a Σ-algebra A. We call B a Σ-subalgebra of A if $\Sigma \cdot B \subset B$. In the same way, one defines Σ-ideals of A. Let $B \subset A$ be a Σ-subalgebra and let $X \subset B$ be a subset. We say that B is Σ-*generated by* X if B is generated by $\Sigma \cdot X$ as a K-subalgebra. In other words, B coincides with the smallest Σ-subalgebra of A containing X. In the same way, one defines Σ-generation for Σ-ideals. Let A, B be Σ-algebras and let $\varphi : A \to B$ be a K-algebra homomorphism. We call φ a Σ-*homomorphism* if $\varphi(\sigma \cdot x) = \sigma \cdot \varphi(x)$ for all $\sigma \in \Sigma, x \in A$.

In the category of Σ-algebras one can define free objects as follows. Let X be a set and denote by $x(\sigma)$ each element (x, σ) of the product set $X(\Sigma) = X \times \Sigma$. Define $P = K[X(\Sigma)]$ the polynomial algebra in all commuting variables $x(\sigma) \in X(\Sigma)$. For any element $\sigma \in \Sigma$ consider the Σ-algebra endomorphism $\bar{\sigma} : P \to P$ such that $x(\tau) \mapsto x(\sigma\tau)$, for all $x(\tau) \in X(\Sigma)$. Then, one has a faithful monoid representation $\rho : \Sigma \to \mathrm{End}(P)$ such that $\rho(\sigma) = \bar{\sigma}$ ad hence P is a Σ-algebra. Note that if Σ is a left-cancellative monoid then all maps $\rho(\sigma)$ are injective.

Proposition 2.1. *Let A be a Σ-algebra and let $f : X \to A$ be any map. Then, a unique Σ-algebra homomorphism $\varphi : P \to A$ is given such that $\varphi(x(1)) = f(x)$, for all $x \in X$.*

Proof. A K-algebra homomorphism $\varphi : P \to A$ is clearly defined by putting $\varphi(x(\sigma)) = \sigma \cdot f(x)$, for any $x \in X$ and $\sigma \in \Sigma$. Then, one has that $\varphi(\tau \cdot \alpha x(\sigma)) = \varphi((\tau \cdot \alpha)x(\tau\sigma)) = (\tau \cdot \alpha)\varphi(x(\tau\sigma)) = (\tau \cdot \alpha)(\tau\sigma \cdot f(x)) = \tau \cdot (\alpha(\sigma \cdot f(x))) = \tau \cdot (\alpha\varphi(x(\sigma))) = \tau \cdot \varphi(\alpha x(\sigma))$, for all $\alpha \in K$, $x \in X$ and $\sigma, \tau \in \Sigma$. $\qquad\square$

Definition 2.2. *We call $P = K[X(\Sigma)]$ the* free Σ-algebra generated by X. *In fact, P is Σ-generated by the subset $X(1) = \{x_i(1) \mid x_i \in X\}$.*

Then, if A is any Σ-algebra which is Σ-generated by X one has that A is isomorphic to the quotient P/J where $J \subset P$ is the Σ-ideal containing all Σ-algebra relations satisfied by the elements of X. In other words, there is a surjective Σ-algebra homomorphism $\varphi : P \to A$ such that $x(1) \mapsto x$ $(x \in X)$ and one defines $J = \mathrm{Ker}\,\varphi$.

From now on, we specialize our assumptions in the following way. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let Σ be a free commutative monoid generated by a finite set, say $\{\sigma_1, \ldots, \sigma_r\}$. Note that $(\Sigma, \cdot)$ is a cancellative monoid isomorphic to $(\mathbb{N}^r, +)$. Under the above assumptions, the finitely generated free Σ-algebra $P = K[X(\Sigma)]$ is called the *algebra of partial difference polynomials*. In literature one finds also the notation $P = K\{X\}$ which emphasizes the role of X as (free) Σ-generating set of the algebra P. If K is a *field of constants* that is Σ acts trivially on K then the difference polynomials of P are *with constant coefficients*. Moreover, one uses the term *ordinary difference* when $r = 1$. The motivation for these names is the following. Let K be a field of functions in the variables $t_1, \ldots, t_r$ and fix $h_1, \ldots, h_r$ some parameters (mesh steps). Assume we may define the action of Σ on K by putting $\sigma \cdot f(t_1, \ldots, t_r) = f(t_1 + \alpha_1 h_1, \ldots, t_r + \alpha_r h_r) \in K$ for all $\sigma = \prod_i \sigma_i^{\alpha_i} \in \Sigma$ and for any function $f \in K$. For instance, one can consider the field of rational functions $K = F(t_1, \ldots, t_k)$ over some field F and $h_1, \ldots, h_r \in F$. Then, in the formal (symbolic) theory of difference equations the variables $x_i(1)$ are by definition unknown functions $u_i(t_1, \ldots, t_r)$ that are considered algebraically independent together with the shifted functions $x_i(\sigma) = \sigma \cdot x_i(1) = u_i(t_1 + \alpha_1 h_1, \ldots, t_r + \alpha_r h_r)$. Then, a Σ-ideal $I \subset P$ is also called a *partial difference ideal* and a Σ-basis of I corresponds to a system of partial difference equations. Since Σ is an infinite set, we have that $P = K[X(\Sigma)]$ is not a Noetherian ring and hence such ideals have bases or Σ-bases which are generally infinite.

3. Gröbner bases of difference ideals

In this section we introduce a Gröbner basis theory for the algebra of partial difference polynomials by extending what has be done in [18] for the case of constant

coefficients. Note that the concept of difference Gröbner basis has arisen also in [10, 13, 17]. We start by noting that the Higman's Lemma [14] implies that the polynomial algebra P can be endowed with a monomial ordering even if the variables set $X(\Sigma)$ is infinite. Denote by $M = \mathrm{Mon}(P)$ the set of all monomials of P.

Proposition 3.1. *Let $\prec$ be a total ordering on M such that*

(i) $1 \prec m$, *for all $m \in M, m \neq 1$;*
(ii) *if $m \prec n$ then $tm \prec tn$, for any $m, n, t \in M$;*
(iii) $\prec$ *is a well-ordering on $X(\Sigma) \subset M$.*

Then, $\prec$ is a well-ordering also on M that is $\prec$ is a monomial ordering on P.

Note that the monoid Σ stabilizes the monomials set M. Then, it is clear that one needs monomial orderings that are compatible with the action of Σ on M.

Definition 3.2. *Let $\prec$ be a monomial ordering of P. We call $\prec$ a monomial Σ-ordering of P if $m \prec n$ implies that $\sigma \cdot m \prec \sigma \cdot n$, for all $m, n \in M$ and $\sigma \in \Sigma$.*

Note that if $\prec$ is a monomial Σ-ordering of P then one has immediately that $\sigma \cdot m \succeq m$, for all $m \in M$ and $\sigma \in \Sigma$. Examples of such orderings can be easily constructed in the following way. Let $Q = K[\sigma_1, \ldots, \sigma_r]$ be the polynomial algebra in the variables σ_j and therefore $\Sigma = \mathrm{Mon}(Q)$. Moreover, let $K[X] = K[x_1, \ldots, x_n]$ be the polynomial algebra in the variables x_i. Fix a monomial ordering $<$ for Q and a monomial ordering $\prec$ for $K[X]$. For any $\sigma \in \Sigma$ denote $X(\sigma) = \{x_i(\sigma) \mid x_i \in X\}$. Clearly $P(\sigma) = K[X(\sigma)]$ is a subalgebra of P that is isomorphic to $K[X]$ and hence it can be endowed with the monomial ordering $\prec$. Since $P = \bigotimes_{\sigma \in \Sigma} P(\sigma)$ one can define a block monomial ordering for P obtained by $<$ and $\prec$.

Proposition 3.3. *Let $m = m(\delta_1) \cdots m(\delta_k), n = n(\delta_1) \cdots n(\delta_k) \in M$ where $m(\delta_i)$, $n(\delta_i) \in M(\delta_i) = \mathrm{Mon}(P(\delta_i))$ and $\delta_1 > \ldots > \delta_k$. We define $m \prec' n$ if and only if there is $1 \leq i \leq k$ such that $m(\delta_j) = n(\delta_j)$ when $j < i$ and $m(\delta_i) \prec n(\delta_i)$. Then, $\prec'$ is a monomial Σ-ordering of P.*

Proof. For all $\sigma \in \Sigma$, one has that $\sigma \cdot m = m(\sigma\delta_1) \cdots m(\sigma\delta_k)$ where $m(\sigma\delta_i) = \sigma \cdot m(\delta_i) \in M(\sigma\delta_i)$ and $\sigma\delta_1 > \ldots > \sigma\delta_k$ since $<$ is a monomial ordering of Q. Assume $m \prec' n$ that is $m(\delta_j) = n(\delta_j)$ for $j < i$ and $m(\delta_i) \prec n(\delta_i)$. Clearly, one has also that $m(\sigma\delta_j) = n(\sigma\delta_j)$. Moreover, by definition of the monomial ordering $\prec$ on all subalgebras $P(\sigma) \subset P$ we have that $m(\delta_i) \prec n(\delta_i)$ if and only if $m(\sigma\delta_i) \prec n(\sigma\delta_i)$. We conclude that $\sigma \cdot m \prec' \sigma \cdot n$. □

From now on, we assume that P is endowed with a monomial Σ-ordering $\prec$. Let $f = \sum_i c_i m_i \in P$ with $m_i \in M$ and $0 \neq c_i \in K$. We denote as usual $\mathrm{lm}(f) = m_k = \max_\prec\{m_i\}$, $\mathrm{lc}(f) = c_k$ and $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lm}(f)$. Since $\prec$ is a Σ-ordering, one has that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f)$ and therefore $\mathrm{lc}(\sigma \cdot f) = \sigma \cdot \mathrm{lc}(f), \mathrm{lt}(\sigma \cdot f) = \sigma \cdot \mathrm{lt}(f)$, for all $\sigma \in \Sigma$. If $G \subset P$ we put $\mathrm{lm}(G) = \{\mathrm{lm}(f) \mid f \in G, f \neq 0\}$ and we denote as $\mathrm{LM}(G)$ the ideal of P generated by $\mathrm{lm}(G)$.

Proposition 3.4. *Let $G \subset P$. Then $\mathrm{lm}(\Sigma \cdot G) = \Sigma \cdot \mathrm{lm}(G)$. In particular, if I is a Σ-ideal of P then $\mathrm{LM}(I)$ is also Σ-ideal.*

Proof. Since P is endowed with a Σ-ordering, one has that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f)$ for any $f \in P, f \neq 0$ and $\sigma \in \Sigma$. Then, $\Sigma \cdot \mathrm{lm}(I) = \mathrm{lm}(\Sigma \cdot I) \subset \mathrm{lm}(I)$ and therefore $\mathrm{LM}(I) = \langle \mathrm{lm}(I) \rangle$ is a Σ-ideal. □

Definition 3.5. *Let* $I \subset P$ *be a* Σ-*ideal and* $G \subset I$. *We call* G *a* Gröbner Σ-*basis of* I *if* $\mathrm{lm}(G)$ *is a* Σ-*basis of* $\mathrm{LM}(I)$. *In other words,* $\Sigma \cdot G$ *is a Gröbner basis of* I *as an ideal of* P.

Since P is not a Noetherian algebra, in general its Σ-ideals have infinite (Gröbner) Σ-bases. Note that one has a similar situation for the free associative algebra and its ideals and this case is strictly related with the algebra of ordinary difference polynomials owing to the notion of "letterplace correspondence" [16, 17, 19]. Nevertheless, in Section 6 we will prove the existence of a class of Σ-ideals containing finite Gröbner Σ-bases. According to [10, 13], such finite bases are also called "difference Gröbner bases".

Let $f, g \in P, f, g \neq 0$ and put $\mathrm{lt}(f) = cm, \mathrm{lt}(g) = dn$ with $m, n \in M$ and $c, d \in K$. If $l = \mathrm{lcm}(m, n)$ one defines the *S-polynomial* $\mathrm{spoly}(f, g) = (l/cm)f - (l/dn)g$.

Proposition 3.6. *For all* $f, g \in P, f, g \neq 0$ *and for any* $\sigma \in \Sigma$ *one has that* $\sigma \cdot \mathrm{spoly}(f, g) = \mathrm{spoly}(\sigma \cdot f, \sigma \cdot g)$.

Proof. Note that $\mathrm{lt}(\sigma \cdot f) = (\sigma \cdot c)(\sigma \cdot m), \mathrm{lt}(\sigma \cdot g) = (\sigma \cdot d)(\sigma \cdot n)$ with $\sigma \cdot m, \sigma \cdot n \in M$ and $\sigma \cdot c, \sigma \cdot d \in K$. Since Σ acts on the variables set $X(\Sigma)$ by injective maps, if $l = \mathrm{lcm}(m, n)$ then $\sigma \cdot l = \mathrm{lcm}(\sigma \cdot m, \sigma \cdot n)$ and therefore we have

$$\sigma \cdot \mathrm{spoly}(f, g) = \sigma \cdot (\tfrac{l}{cm}f - \tfrac{l}{dn}g) =$$
$$\tfrac{\sigma \cdot l}{(\sigma \cdot c)(\sigma \cdot m)}\sigma \cdot f - \tfrac{\sigma \cdot l}{(\sigma \cdot d)(\sigma \cdot n)}\sigma \cdot g = \mathrm{spoly}(\sigma \cdot f, \sigma \cdot g).$$

$\square$

In the theory of Gröbner bases one has the following important notion.

Definition 3.7. *Let* $f \in P, f \neq 0$ *and* $G \subset P$. *If* $f = \sum_i f_i g_i$ *with* $f_i \in P, g_i \in G$ *and* $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i)\mathrm{lm}(g_i)$ *for all* i, *we say that* f *has a Gröbner representation with respect to* G.

Note that if $f = \sum_i f_i g_i$ is a Gröbner representation then $\sigma \cdot f = \sum_i (\sigma \cdot f_i)(\sigma \cdot g_i)$ is also a Gröbner representation, for any $\sigma \in \Sigma$. In fact, from $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i)\mathrm{lm}(g_i)$ it follows that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f) \succeq (\sigma \cdot \mathrm{lm}(f_i))(\sigma \cdot \mathrm{lm}(g_i)) = \mathrm{lm}(\sigma \cdot f_i)\mathrm{lm}(\sigma \cdot g_i)$, for all indices i. For the Gröbner Σ-bases of P we have hence the following characterization.

Proposition 3.8 (Σ-criterion)**.** *Let* G *be a* Σ-*basis of a* Σ-*ideal* $I \subset P$. *Then,* G *is a Gröbner* Σ-*basis of* I *if and only if for all* $f, g \in G, f, g \neq 0$ *and for any* $\sigma, \tau \in \Sigma$ *such that* $\gcd(\sigma, \tau) = 1$ *and* $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$, *the S-polynomial* $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ *has a Gröbner representation with respect to* $\Sigma \cdot G$.

Proof. Recall that G is a Gröbner Σ-basis if and only if $\Sigma \cdot G$ is a Gröbner basis of I. By the Buchberger's criterion [4] or by the Bergman's diamond lemma [2] this happens if and only if the S-polynomials $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ have a Gröbner representation with respect to $\Sigma \cdot G$, for all $f, g \in G, f, g \neq 0$ and $\sigma, \tau \in \Sigma$. By the product criterion we may restrict ourselves to considering only S-polynomials such that $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ since $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f), \mathrm{lm}(\tau \cdot g) = \tau \cdot \mathrm{lm}(g)$. Then, let $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ be any such S-polynomial and put $\delta = \gcd(\sigma, \tau)$ and therefore $\sigma = \delta \sigma', \tau = \delta \tau'$ with $\sigma', \tau' \in \Sigma, \gcd(\sigma', \tau') = 1$. One has that $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot \mathrm{spoly}(\sigma' \cdot f, \tau' \cdot g)$ owing to Proposition 3.6. Note now that if $\mathrm{spoly}(\sigma' \cdot f, \tau' \cdot g) = h = \sum_\nu f_\nu (\nu \cdot g_\nu)$ ($\nu \in \Sigma, f_\nu \in P, g_\nu \in G$) is a Gröbner representation with respect to $\Sigma \cdot G$ then also $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot h = \sum_\nu (\delta \cdot f_\nu)(\delta \nu \cdot g_\nu)$ is a Gröbner

representation because $\prec$ is a Σ-ordering of P. We conclude that the S-polynomials to be checked for Gröbner representations may be restricted to the ones satisfying both the conditions $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ and $\gcd(\sigma, \tau) = 1$. $\square$

The above result implies immediately the correctness of the following algorithm for the computation of Gröbner Σ-bases.

Algorithm 3.1 SigmaGBasis

Input: H, a Σ-basis of a Σ-ideal $I \subset P$.
Output: G, a Gröbner Σ-basis of I.
$G := H$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ **do**
    choose $(f, g) \in B$;
    $B := B \setminus \{(f, g)\}$;
    for all $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1, \gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ **do**
        $h := \mathrm{REDUCE}(\mathrm{spoly}(\sigma \cdot f, \tau \cdot g), \Sigma \cdot G)$;
        if $h \neq 0$ **then**
            $B := B \cup \{(g, h), (h, h) \mid g \in G\}$;
            $G := G \cup \{h\}$;
        end if;
    end for;
end while;
return G.

Note that the function REDUCE is given by the following standard routine.

Algorithm 3.2 REDUCE

Input: $G \subset P$ and $f \in P$.
Output: $h \in P$ such that $f - h \in \langle G \rangle$ and $h = 0$ or $\mathrm{lm}(h) \notin \mathrm{LM}(G)$.
$h := f$;
while $h \neq 0$ and $\mathrm{lm}(h) \in \mathrm{LM}(G)$ **do**
    choose $g \in G, g \neq 0$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(h)$;
    $h := h - (\mathrm{lt}(h)/\mathrm{lt}(g))g$;
end while;
return h.

Clearly, the chain criterion can be added to SigmaGBasis to shorten the number of S-polynomials that have to be reduced. In fact, one can view this algorithm as a variant of the Buchberger's procedure applied to the basis $\Sigma \cdot H$, where Proposition 3.8 provides the additional criterion "$\gcd(\sigma, \tau) = 1$" to avoid useless pairs. Note also that for any pair of elements $f, g \in G$ and for all $\sigma, \tau \in \Sigma$ there are only a finite number of S-polynomials $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ satisfying both the criteria $\gcd(\sigma, \tau) = 1$ and $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$. A proof is given by the arguments contained in Proposition 4.7.

It is important to remark that we do not have general termination for the algorithm SigmaGBasis owing to Non-Noetherianity of the ring P. In fact, even if a Σ-ideal $I \subset P$ has a finite Σ-basis this may be not true for its initial Σ-ideal $\mathrm{LM}(I)$

that is I may have no finite Gröbner Σ-basis. In the following sections we propose then two possible solutions to this problem. First, we introduce a suitable grading on P that is compatible with the action of Σ which implies that the truncated version of the algorithm SigmaGBasis with homogeneous input stops in a finite number of steps. Another approach consists in obtaining finite Gröbner Σ-bases when suitable elements belong to the given Σ-ideal I. More precisely, we prove the Noetherian property for suitable (quotient) Σ-algebras P/I.

4. Grading and truncation

A useful grading for the free Σ-algebra P can be introduced in the following way. Consider the set $\hat{\mathbb{N}} = \mathbb{N} \cup \{-\infty\}$ endowed with the binary operations max and $+$. Then $(\hat{\mathbb{N}}, \max, +)$ is clearly a commutative idempotent semiring. Moreover, for any $\sigma = \prod_i \sigma_i^{\alpha_i} \in \Sigma$ we put $\deg(\sigma) = \sum_i \alpha_i$.

Definition 4.1. *Let* $\mathrm{ord} : M \to \hat{\mathbb{N}}$ *be the unique mapping such that*
   (i) $\mathrm{ord}(1) = -\infty$;
   (ii) $\mathrm{ord}(mn) = \max(\mathrm{ord}(m), \mathrm{ord}(n))$, *for any* $m, n \in M$;
   (iii) $\mathrm{ord}(x_i(\sigma)) = \deg(\sigma)$, *for all* $i \geq 0$ *and* $\sigma \in \Sigma$.
Then, the map ord *is a monoid homomorphism from* $(M, \cdot)$ *to* $(\hat{\mathbb{N}}, \max)$. *We call* ord *the* order function *of* P.

For any monomial $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k}$ different from 1 we have that $\mathrm{ord}(m) = \max(\deg(\delta_1), \ldots, \deg(\delta_k))$. If $P_d = \langle m \in M \mid \mathrm{ord}(m) = d \rangle_K \subset P$ then $P = \bigoplus_{d \in \hat{\mathbb{N}}} P_d$ is clearly a grading of the algebra P over the commutative monoid $(\hat{\mathbb{N}}, \max)$.

Proposition 4.2. *The following properties hold for the order function:*
   (i) $\mathrm{ord}(\sigma \cdot m) = \deg(\sigma) + \mathrm{ord}(m)$, *for any* $\sigma \in \Sigma$ *and* $m \in M$;
   (ii) $\mathrm{ord}(\mathrm{lcm}(m, n)) = \mathrm{ord}(mn) = \max(\mathrm{ord}(m), \mathrm{ord}(n))$, *for all* $m, n \in M$. *Therefore, if* $m \mid n$ *then* $\mathrm{ord}(m) \leq \mathrm{ord}(n)$.

Proof. If $m = 1$ then $\mathrm{ord}(\sigma \cdot m) = \mathrm{ord}(m) = -\infty = \deg(\sigma) + \mathrm{ord}(m)$. If otherwise $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k}$ then $\sigma \cdot m = x_{i_1}(\sigma\delta_1)^{\alpha_1} \cdots x_{i_k}(\sigma\delta_k)^{\alpha_k}$ and hence $\mathrm{ord}(\sigma \cdot m) = \max(\deg(\sigma\delta_1), \ldots, \deg(\sigma\delta_k)) = \deg(\sigma) + \max(\deg(\delta_1), \ldots, \deg(\delta_k)) = \deg(\sigma) + \mathrm{ord}(m)$. To prove (ii) it is sufficient to note that the order of a monomial does not depend on the exponents of the variables occurring in it. $\square$

Definition 4.3. *An ideal* $I \subset P$ *is called* ord-graded *if* $I = \sum_d I_d$ *with* $I_d = I \cap P_d$. *If* I *is also a* Σ-ideal *then* $\sigma \cdot I_d \subset I_{\deg(\sigma)+d}$ *for any* $\sigma \in \Sigma$ *and* $d \in \hat{\mathbb{N}}$.

Let $f, g \in P, f \neq g$ be any pair of ord-homogeneous elements. Then, the S-polynomial $h = \mathrm{spoly}(f, g)$ is also ord-homogeneous and we have that $\mathrm{ord}(h) = \max(\mathrm{ord}(f), \mathrm{ord}(g))$. If $\mathrm{ord}(f), \mathrm{ord}(g) \leq d$ for some $d \in \mathbb{N}$, one has therefore that $\mathrm{ord}(h) \leq d$ which implies the following result.

Proposition 4.4 (Termination by truncation)**.** *Let* $I \subset P$ *be a* ord-*graded* Σ-ideal *and let* $d \in \mathbb{N}$. *Assume there is a* ord-*homogeneous basis* $H \subset I$ *such that* $H_d = \{f \in H \mid \mathrm{ord}(f) \leq d\}$ *is a finite set. Then, there exists also a* ord-*homogeneous Gröbner* Σ-basis G *of* I *such that* G_d *is a finite set. In other words, if one uses for* SigmaGBasis *a selection strategy of the S-polynomials based on their orders then the d-truncated version of* SigmaGBasis *terminates in a finite number of steps.*

Proof. In the algorithm SIGMAGBASIS one computes a subset G of a Gröbner basis $G' = \Sigma \cdot G$ obtained by applying the Buchberger's algorithm to the basis $H' = \Sigma \cdot H$ of the ideal I. Moreover, Proposition 4.2 implies that the set H' and hence G' consists of ord-homogeneous elements. Define $H'_d = \{\sigma \cdot f \mid \sigma \in \Sigma, f \in H, \deg(\sigma) + \mathrm{ord}(f) \leq d\}$. Since $\Sigma_d = \{\sigma \in \Sigma \mid \deg(\sigma) \leq d\}$ and by hypothesis H_d are finite sets, we have that $H'_d \subset \Sigma_d \cdot H_d$ is also a finite set. Denote now Y_d the finite set of variables of P occurring in the elements of H'_d and define the subalgebra $P_{(d)} = K[Y_d] \subset P$. In fact, the d-truncated algorithm SIGMAGBASIS computes a subset of a Gröbner basis of the ideal $I_{(d)} \subset P_{(d)}$ generated by H'_d. The Noetherianity of the finitely generated polynomial ring $P_{(d)}$ provides then termination. $\square$

Note that this result implies algorithmic solution to the ideal membership for finitely generated ord-graded Σ-ideals. Another consequence of the grading defined by the order function is that one has a criterion, also in the non-graded case, for verifying that a Σ-basis computed by the algorithm SIGMAGBASIS using a finite number of variables of P is a complete finite Gröbner Σ-basis, whenever this basis exists. This is of course important because actual computations can be only performed over a finite number of variables.

Definition 4.5. *Let $\prec$ be a monomial Σ-ordering of P. We say that $\prec$ is compatible with the order function if $\mathrm{ord}(m) < \mathrm{ord}(n)$ implies that $m \prec n$, for all $m, n \in M$.*

Proposition 4.6. *Denote by $\prec$ the monomial Σ-ordering of P defined in Proposition 3.3 and let $<$ be the monomial ordering of $Q = K[\sigma_1, \ldots, \sigma_r]$ that is used to define $\prec$. If $<$ is compatible with function $\deg$ then $\prec$ is compatible with function ord.*

Proof. Let $m = m(\delta_1) \cdots m(\delta_k), n = n(\delta_1) \cdots n(\delta_k)$ two monomials of P with $m(\delta_i), n(\delta_i) \in M(\delta_i)$ and $\delta_1 > \ldots > \delta_k$ (hence $\deg(\delta_1) \geq \ldots \geq \deg(\delta_1)$). Assume $m \prec n$ that is there is $1 \leq i \leq k$ such that $m(\delta_j) = n(\delta_j)$ when $j < i$ and $m(\delta_i) \prec n(\delta_i)$. If $i > 1$ or $m(\delta_i) \neq 1$ one has clearly $\mathrm{ord}(m) = \mathrm{ord}(n) = \deg(\delta_1)$. Otherwise, we conclude that $\mathrm{ord}(m) \leq \deg(\delta_1) = \mathrm{ord}(n)$. $\square$

As before, we denote $\Sigma_d = \{\sigma \in \Sigma \mid \deg(\sigma) \leq d\}$.

Proposition 4.7 (Finite Σ-criterion)**.** *Assume P be endowed with a monomial Σ-ordering compatible with the order function. Let $G \subset P$ be a finite set and define the Σ-ideal $I = \langle G \rangle_\Sigma$. Moreover, denote $d = \max\{\mathrm{ord}(\mathrm{lm}(g)) \mid g \in G, g \neq 0\}$. Then, G is a Gröbner Σ-basis of I if and only if for all $f, g \in G$ and for any $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1$ and $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$, the S-polynomial $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ has a Gröbner representation with respect to the finite set $\Sigma_{2d} \cdot G$.*

Proof. Let $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = h = \sum_\nu f_\nu(\nu \cdot g_\nu)$ be a Gröbner representation with respect to $\Sigma \cdot G$ that is $\mathrm{lm}(h) \succeq \mathrm{lm}(f_\nu)(\nu \cdot \mathrm{lm}(g_\nu))$ for all ν. We want to bound the degree of the elements $\nu \in \Sigma$ occurring in this representation. Put $m = \mathrm{lm}(f), n = \mathrm{lm}(g)$ and hence $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot m, \mathrm{lm}(\sigma \cdot g) = \sigma \cdot n$. By product criterion, we assume that $u = \gcd(\sigma \cdot m, \tau \cdot n) \neq 1$. Then, there is a variable $x_i(\sigma\alpha) = x_i(\tau\beta)$ that divides u where $x_i(\alpha)$ divides m and hence $\deg(\alpha) \leq \mathrm{ord}(m) \leq d$ and $x_i(\beta)$ divides n and therefore $\deg(\beta) \leq \mathrm{ord}(n) \leq d$. Then $\sigma\alpha = \tau\beta$ and one has that $\sigma \mid \beta, \tau \mid \alpha$ because we assume also that $\gcd(\sigma, \tau) = 1$. We conclude that $\deg(\sigma), \deg(\tau) \leq d$ and if $v =$

$\operatorname{lcm}(\sigma \cdot m, \tau \cdot m)$ then $\operatorname{ord}(v) = \max(\deg(\sigma) + \operatorname{ord}(m), \deg(\tau) + \operatorname{ord}(n)) \leq 2d$. Clearly $v \succ \operatorname{lm}(h) \succeq \nu \cdot \operatorname{lm}(g_\nu)$ and hence $2d \geq \operatorname{ord}(v) \geq \deg(\nu) + \operatorname{ord}(\operatorname{lm}(g_\nu)) \geq \deg(\nu)$. $\square$

Under the assumption of a Σ-ordering compatible with the order function and for Σ-ideals that admit finite Gröbner Σ-bases, by above criterion one has an actual algorithm to compute such a basis in a finite number of steps. In fact, this can be obtained as an adaptative procedure that keeps the bound $2d$ for the degree of the elements of Σ applied to the generators, constantly updated with respect to the maximal order d of the leading monomials of the current generators.

5. An illustrative example

In this section we apply the algorithm SigmaGBasis to an example arising from the discretization of a well-known system of partial differential equations. Consider the unsteady two-dimensional motion of an incompressible viscous liquid of constant viscosity which is governed by the following system

$$\begin{cases} u_x + v_y = 0, \\ u_t + uu_x + vu_y + p_x - \frac{1}{\rho}(u_{xx} + u_{yy}) = 0, \\ v_t + uv_x + vv_y + p_y - \frac{1}{\rho}(v_{xx} + v_{yy}) = 0. \end{cases}$$

The last two nonlinear equations are the Navier-Stokes equations and the first linear equation is the continuity one. Equations are given in the dimensionless form where (u, v) represents the velocity field and the function p is the pressure. The parameter ρ denotes the Reynolds number. For defining a finite difference approximation of this system one has therefore to fix $X = \{u, v, p\}$ and $\Sigma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ since all functions are trivariate ones. To simplify notations, we identify Σ with the additive monoid $\mathbb{N}^3$ and we have that $P = K[X(\Sigma)] = K[u(i, j, k), v(i, j, k), p(i, j, k) \mid i, j, k \geq 0]$. The base field K is the field of rational numbers. The approximation of the derivatives of the function u is given by the following formulas (forward differences)

$$\begin{aligned} u_x &\approx \frac{u(x+h_x,y,t)-u(x,y,t)}{h_x} = \frac{u(1,0,0)-u(0,0,0)}{h_x}, \\ u_y &\approx \frac{u(x,y+h_y,t)-u(x,y,t)}{h_y} = \frac{u(0,1,0)-u(0,0,0)}{h_y}, \\ u_t &\approx \frac{u(x,y,t+h_t)-u(x,y,t)}{h_t} = \frac{u(0,0,1)-u(0,0,0)}{h_t}, \\ u_{xx} &\approx \frac{u(x+2h_x,y,t)-2u(x+h_x,y,t)+u(x,y,t)}{h_x^2} = \frac{u(2,0,0)-2u(1,0,0)+u(0,0,0)}{h_x^2}, \\ u_{yy} &\approx \frac{u(x,y+2h_y,t)-2u(x,y+h_y,t)+u(x,y,t)}{h_y^2} = \frac{u(0,2,0)-2u(0,1,0)+u(0,0,0)}{h_y^2}, \end{aligned}$$

where h_x, h_y, h_t are parameters (mesh steps). One has similar approximations for the derivatives of the functions v, p. Then, the Navier-Stokes system is approximated by the following system of partial difference equations

$$\begin{cases} f_1 = \frac{1}{h_x}(u(1,0,0) - u(0,0,0)) + \frac{1}{h_y}(v(0,1,0) - v(0,0,0)) = 0, \\ f_2 = -\frac{1}{\rho h_x^2}u(2,0,0) - \frac{1}{\rho h_y^2}u(0,2,0) + (\frac{2}{\rho h_x^2} + \frac{1}{h_x}u(0,0,0))u(1,0,0)+ \\ \frac{1}{h_x}p(1,0,0) + (\frac{2}{\rho h_y^2} + \frac{1}{h_y}v(0,0,0))u(0,1,0) + \frac{1}{h_t}u(0,0,1) - \frac{1}{h_x}u(0,0,0)^2 - \\ (\frac{1}{\rho h_x^2} + \frac{1}{\rho h_y^2} + \frac{1}{h_t} + \frac{1}{h_y}v(0,0,0))u(0,0,0) - \frac{1}{h_x}p(0,0,0) = 0, \\ f_3 = -\frac{1}{\rho h_x^2}v(2,0,0) - \frac{1}{\rho h_y^2}v(0,2,0) + (\frac{2}{\rho h_x^2} + \frac{1}{h_x}u(0,0,0))v(1,0,0)+ \\ (\frac{2}{\rho h_y^2} + \frac{1}{h_y}v(0,0,0))v(0,1,0) + \frac{1}{h_y}p(0,1,0) + \frac{1}{h_t}v(0,0,1) - \frac{1}{h_x}u(0,0,0)v(0,0,0) \\ -\frac{1}{h_y}v(0,0,0)^2 - (\frac{1}{\rho h_x^2} + \frac{1}{\rho h_y^2} + \frac{1}{h_t})v(0,0,0) - \frac{1}{h_y}p(0,0,0) = 0. \end{cases}$$

We encode this system as the Σ-ideal $I = \langle f_1, f_2, f_3 \rangle_\Sigma \subset P$ and we want to compute a (hopefully finite) Gröbner Σ-basis of I. For instance, we may want to have such basis to check for "strong-consistency" [10] of the finite difference approximation. We fix the degree reverse lexicographic ordering on the polynomial algebra $K[\sigma_1, \sigma_2, \sigma_3]$ ($\sigma_1 > \sigma_2 > \sigma_3$) and the lexicographic ordering on $K[u, v, p]$ ($u \succ v \succ p$). By Proposition 3.3 one obtains therefore a (block) monomial Σ-ordering for P. Note that this ordering is compatible with the order function and hence Proposition 4.7 is applicable to certify completeness of a Gröbner Σ-basis computed over some finite set of variables $\{u(i, j, k), v(i, j, k), p(i, j, k) \mid i + j + k \leq d\}$.

With respect to the monomial ordering assigned to P, the leading monomials of the Σ-generators of I are $\mathrm{lm}(f_1) = u(1, 0, 0), \mathrm{lm}(f_2) = u(2, 0, 0), \mathrm{lm}(f_3) = v(2, 0, 0)$. Since $\sigma_1 \cdot \mathrm{lm}(f_1) = \mathrm{lm}(f_2)$, by interreducing f_2 with respect to the set $\Sigma \cdot \{f_1, f_3\}$ we obtain the element

$$f_2' = h_y h_t v(1, 1, 0) - h_x h_t u(0, 2, 0) - h_y h_t v(1, 0, 0) + \rho h_y^2 h_t p(1, 0, 0) +$$
$$(2h_x h_t + \rho h_x h_y h_t v(0, 0, 0))u(0, 1, 0) - (h_y h_t + \rho h_x h_y h_t u(0, 0, 0))v(0, 1, 0) +$$
$$\rho h_x h_y^2 u(0, 0, 1) - (h_x h_t + \rho h_x h_y^2)u(0, 0, 0) - \rho h_y^2 h_t p(0, 0, 0) + h_y h_t v(0, 0, 0)$$

whose leading monomial is $\mathrm{lm}(f_2') = v(1, 1, 0)$. Owing to the Σ-criterion, the only S-polynomial to consider is then $\mathrm{spoly}(\sigma_1 \cdot f_2', \sigma_2 \cdot f_3)$ whose reduction with respect to $\Sigma \cdot \{f_1, f_2', f_3\}$ leads to the new element

$$f_4 = h_y^3 h_t p(2, 0, 0) + (-h_x^2 h_y h_t u(0, 0, 0) - h_x^3 h_t v(0, 0, 0) + h_x^3 h_t v(0, 1, 0) +$$
$$h_x^2 h_y h_t u(0, 1, 0))u(0, 2, 0) + (-h_x^2 h_y h_t v(1, 0, 0) + h_x^2 h_y h_t v(0, 1, 0))v(0, 2, 0) +$$
$$h_x^2 h_y h_t p(0, 2, 0) + (2h_x^2 h_y h_t v(0, 1, 0) + 2h_x h_y^2 h_t u(0, 1, 0) - h_x^2 h_y h_t v(0, 0, 0) -$$
$$2h_x h_y^2 h_t u(0, 0, 0))v(1, 0, 0) + (\rho h_x^2 h_y^2 h_t v(0, 0, 0) + \rho h_x h_y^3 h_t u(0, 0, 0) -$$
$$\rho h_x^2 h_y^2 h_t v(0, 1, 0) - \rho h_x h_y^3 h_t u(0, 1, 0) - 2h_y^3 h_t)p(1, 0, 0) - (\rho h_x^2 h_y^2 h_t v(0, 0, 0) +$$
$$2h_x^2 h_y h_t)u(0, 1, 0)^2 + ((\rho h_x^2 h_y^2 h_t u(0, 0, 0) - \rho h_x^3 h_y h_t v(0, 0, 0) - 2h_x^3 h_t)v(0, 1, 0) -$$
$$\rho h_x^2 h_y^3 u(0, 0, 1) + (3h_x^2 h_y h_t + \rho h_x^2 h_y^3 + \rho h_x^2 h_y^2 h_t v(0, 0, 0))u(0, 0, 0) +$$
$$\rho h_x^3 h_y h_t v(0, 0, 0)^2 + (2h_x^3 h_t - 2h_x h_y^2 h_t)v(0, 0, 0) + \rho h_x h_y^3 h_t p(0, 0, 0))u(0, 1, 0) +$$
$$\rho h_x^3 h_y h_t u(0, 0, 0)v(0, 1, 0)^2 + (-\rho h_x^3 h_y^2 u(0, 0, 1) - \rho h_x^2 h_y^2 h_t u(0, 0, 0)^2 +$$
$$(\rho h_x^3 h_y^2 + h_x^3 h_t - \rho h_x^3 h_y h_t v(0, 0, 0))u(0, 0, 0) - 3h_x^2 h_y h_t v(0, 0, 0) +$$
$$\rho h_x^2 h_y^2 h_t p(0, 0, 0))v(0, 1, 0) - 2h_x^2 h_y h_t p(0, 1, 0) + (\rho h_x^3 h_y^2 v(0, 0, 0) +$$
$$\rho h_x^2 h_y^3 u(0, 0, 0))u(0, 0, 1) - (\rho h_x^2 h_y^3 + h_x^2 h_y h_t)u(0, 0, 0)^2 + ((-h_x^3 h_t - \rho h_x^3 h_y^2 +$$
$$2h_x h_y^2 h_t)v(0, 0, 0) - \rho h_x h_y^3 h_t p(0, 0, 0))u(0, 0, 0) + 2h_t h_x^2 h_y h_t v(0, 0, 0)^2 -$$
$$\rho h_x^2 h_y^2 h_t v(0, 0, 0)p(0, 0, 0) + (h_y^3 h_t + h_x^2 h_y h_t)p(0, 0, 0).$$

The leading monomial of this difference polynomial is $\mathrm{lm}(f_4) = p(2, 0, 0)$ and no more S-polynomials are defined. We conclude that the set $\{f_1, f_2', f_3, f_4\}$ is a (finite) Gröbner Σ-basis of the Σ-ideal $I \subset P$. Since we make use of a monomial Σ-ordering for P, this is equivalent to say that $\Sigma \cdot \{f_1, f_2', f_3, f_4\}$ is a Gröbner basis of the ideal I and this can be verified also by applying the usual Gröbner bases routines to the basis $\Sigma \cdot \{f_1, f_2, f_3\}$. Because the maximal order in the input generators is 2, by Proposition 4.7 it is reasonable to bound initially the order of the variables of P to 4 or 5. Note that the computing time for obtaining a Gröbner basis of I with implementation in Maple of the Faugere's F4 algorithm amounts to 20 seconds for order 4 and 5 hours for order 5 on our laptop Intel Core 2 Duo at 2.10 GHz with 8 GB RAM. By the algorithm SIGMAGBASIS that we implemented in the Maple language as a variant of the Buchberger's algorithm (see [18]), the computing time for a Gröbner Σ-basis of I is instead 0 seconds for order 4 and 3 seconds for order 5 since just two reductions are needed.

6. A NOETHERIANITY CRITERION

As already noted, a critical feature of the algebra of partial difference polynomials $P = K[X(\Sigma)]$ is that some of its Σ-ideals are not only infinitely generated but also infinitely Σ-generated. One finds an immediate counterexample for $\Sigma = \langle \sigma \rangle$ that we identify with the additive monoid $\mathbb{N}$. Then, the ideal $I = \langle x_i(0)x_i(1), x_i(0)x_i(2), \ldots \rangle_\Sigma$ has clearly no finite $\mathbb{N}$-basis. It is interesting to note that if one considers a larger semigroup like $\text{Inc}(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{N} \mid f$ strictly increasing$\}$ acting on $P = K[X(\mathbb{N})]$ as $f \cdot x_i(j) = x_i(f(j))$, one obtains that P is $\text{Inc}(\mathbb{N})$-Noetherian [1]. We may say hence that the monoid $\mathbb{N}$ is "too small" to provide $\mathbb{N}$-Noetherianity. One way to solve this problem is to consider suitable quotients of the algebra of partial difference polynomials where Noetherianity is restored. A similar approach is used for the free associative algebra which is also Non-Noetherian where the concepts of "algebras of solvable type, PBW algebras, G-algebras", etc naturally arise (see for instance [20]).

We start now with a general discussion for (commutative) algebras generated by a countable set of elements. Let $Y = \{y_1, y_2, \ldots\}$ be a countable set and denote $P = K[Y]$ the polynomial algebra with variables set Y. Since P is a free algebra, all algebras generated by a countable set of elements are clearly isomorphic to quotients $P' = P/J$, where J is some ideal of P. To control the cosets in P', a standard approach consists in defining a normal form modulo J associated to a monomial ordering of P. Then, let $\prec$ be a monomial ordering of P such that $y_1 \prec y_2 \prec \ldots$.

Definition 6.1. *Put $M = \text{Mon}(P)$ and denote $M'' = M \setminus \text{lm}(J)$. Moreover, define the subspace $P'' = \langle M'' \rangle_K \subset P$. The elements of M'' are called* normal monomials *modulo J (with respect to $\prec$). The polynomials in P'' are said* in normal form *modulo J.*

Since P is endowed with a monomial ordering, by a standard argument based on the procedure REDUCE applied for the set J one obtains the following result.

Proposition 6.2. *A K-linear basis of the algebra P' is given by the set $M' = \{m + J \mid m \in M''\}$.*

Definition 6.3. *Let $f \in P$. Denote $\text{NF}(f)$ the unique element of P'' such that $f - \text{NF}(f) \in J$. In other words, one has $\text{NF}(f) = \text{REDUCE}(f, J)$. We call $\text{NF}(f)$ the* normal form *of f modulo J (with respect to $\prec$).*

By Proposition 6.2, one has that the mapping $f + J \mapsto NF(f)$ defines a linear isomorphism between $P' = P/J$ and $P'' = \langle M'' \rangle_K$. An algebra structure is defined hence for P'' by imposing that such mapping is also an algebra isomorphism that is we define $f \cdot g = \text{NF}(fg)$, for all $f, g \in P''$. By means of this, one has a complete identification of M', M'' and P', P'' that we will use from now on. We define then the set of *normal variables*

$$Y' = Y \cap M' = Y \setminus \text{lm}(J).$$

Clearly, normal variables depend strictly on the monomial ordering one uses in P.

Proposition 6.4 (Noetherianity criterion). *Let P be endowed with a monomial ordering. If the set of normal variables Y' is finite then P' is a Noetherian algebra.*

Proof. It is sufficient to note that all normal monomials are product of normal variables and therefore the quotient algebra $P' = P/J$ is in fact generated by the

set Y'. If Y' is finite then P' is a finitely generated (commutative) algebra and hence it satisfies the Noetherian property. □

We need now to introduce the notion of Gröbner basis for the ideals of $P' = P/J$. After identification of cosets with normal forms, recall that $M' = M \setminus \mathrm{lm}(J)$ and $P' = \langle M' \rangle_K$ is a subspace of P endowed with multiplication $f \cdot g = \mathrm{NF}(fg)$, for all $f, g \in P'$. Then, all ideals $I' \subset P'$ have the form $I' = I/J = \{\mathrm{NF}(f) \mid f \in I\}$, for some ideal $J \subset I \subset P$. Note that $\mathrm{NF}(f) \in I$ for any $f \in I$, which implies that in fact $I' = I \cap P'$. Since the quotient algebra P'/I' is isomorphic to P/I and Gröbner bases give rise to K-linear bases of normal monomials for the quotients, one introduces the following definition.

Definition 6.5. *Let $I' = I \cap P'$ be an ideal of P' where $I \supset J$ is an ideal of P. Moreover, let $G' \subset I'$. We call G' a* Gröbner basis *of I' if $G' \cup J$ is a Gröbner basis of I.*

Let $G \subset P$. Recall that $\mathrm{LM}(G)$ denotes the ideal of P generated by the set $\mathrm{lm}(G) = \{\mathrm{lm}(g) \mid g \in G, g \neq 0\}$.

Proposition 6.6. *Let I' be an ideal of P' and let $G' \subset I'$. Then, the set G' is a Gröbner basis of I' if and only if $\mathrm{LM}(G') = \mathrm{LM}(I')$.*

Proof. Assume $\mathrm{LM}(G') = \mathrm{LM}(I')$. Let $f \in I$ and denote $f' = \mathrm{NF}(f)$. If $\mathrm{lm}(f) \notin \mathrm{LM}(J)$ then $\mathrm{lm}(f) = \mathrm{lm}(f')$. Since $\mathrm{lm}(f') \in \mathrm{LM}(I') \subset \mathrm{LM}(G')$ then $\mathrm{lm}(f) = \mathrm{lm}(f') = m\,\mathrm{lm}(g')$, for some $m \in M, g' \in G'$. We conclude that $G' \cup J$ is a Gröbner basis of I. Suppose now that the latter condition holds. Since $G' \subset I'$ we have clearly that $\mathrm{LM}(G') \subset \mathrm{LM}(I')$. Let now $f' \in I' \subset I$. Then, there is $m \in M, g' \in G' \cup J$ such that $\mathrm{lm}(f') = m\,\mathrm{lm}(g)$. Since $\mathrm{lm}(f') \in M'$ then also $\mathrm{lm}(g) \in M'$ and hence $g \in G'$. □

Proposition 6.7. *Assume that the set of normal variables $Y' = Y \cap M'$ is finite. Then, any monomial ideal $I = \langle I \cap M' \rangle \subset P$ has a finite basis.*

Proof. It is sufficient to invoke the Dickson's Lemma (see for instance [8]) for the ideal I which is generated by normal monomials that are products of a finite number of normal variables. □

Corollary 6.8. *If Y' is a finite set then any ideal $I' \subset P'$ has a finite Gröbner basis.*

Proof. According to Proposition 6.6, consider the ideal $\mathrm{LM}(I') \subset P$ which is generated by the set of normal monomials $\mathrm{lm}(I')$. Then, it is sufficient to apply Proposition 6.7 to this ideal. □

It is clear that if G is any Gröbner basis of an ideal $J \neq P$ then $Y' = Y \setminus \mathrm{lm}(G)$. Since Y is a countable set, if Y' is a finite set and hence $P' = K[Y']$ is a Noetherian algebra then G needs to be an infinite set. In general, such Gröbner basis cannot be computed but this is possible when P is a Σ-algebra and J is a Σ-ideal. From now on, we assume again that $P = K[X(\Sigma)]$ is the algebra of partial difference polynomials. Let $J \subset P$ be any Σ-ideal that is $P' = P/J$ is any Σ-algebra which is (finitely) Σ-generated by $X(1)$ and J is the ideal of Σ-algebra relations satisfied by such generating set. Let P be endowed with a monomial Σ-ordering $\prec$ and define, as before, the set $M' \subset M = \mathrm{Mon}(P)$ of all normal monomials and the set $X(\Sigma)' = X(\Sigma) \cap M'$ of all normal variables. Then P' is an algebra generated by

$X(\Sigma)'$. More precisely, the Σ-algebra P' is Σ-generated by $X(1)' = X(1) \cap M'$. In fact, the set $X(\Sigma) \setminus X(\Sigma)' = X(\Sigma) \cap \mathrm{lm}(J)$ is clearly stable under the action of Σ that is if $x_i(\sigma) \in X(\Sigma)'$ then $x_i(1) \in X(1)'$. The structure of Σ-algebra of P' is given by defining, for all $\sigma \in \Sigma$ and $x_i(1) \in X(1)'$

$$\sigma \cdot x_i(1) = \mathrm{NF}(x_i(\sigma)).$$

Proposition 6.9 (Finiteness criterion). *The set of normal variables $X(\Sigma)'$ is finite if and only if for all $1 \leq i \leq n, 1 \leq j \leq r$ one has that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$, for some integers $d_{ij} \geq 0$.*

Proof. Put $x_i(\Sigma) = \{x_i(\sigma) \mid \sigma \in \Sigma\}$ and denote $x_i(\Sigma)' = x_i(\Sigma) \cap X(\Sigma)'$, for any $i = 1, 2, \ldots, n$. We have then to characterize when $x_i(\Sigma)'$ is a finite set. Consider the polynomial algebra $Q = K[\sigma_1, \ldots, \sigma_r]$ and a monomial ideal $I \subset Q$. It is well-known that the quotient algebra Q/I is finite dimensional if and only if there are integers $d_j \geq 0$ such that $\sigma_j^{d_j} \in I$, for all $j = 1, 2, \ldots, r$. In other words, $x_i(\Sigma)'$ is a finite set if and only if there exist integers $d_{ij} \geq 0$ such that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$, for all indices i, j. $\qquad \square$

Corollary 6.10 (Termination by membership). *Let $J \subset P$ be a Σ-ideal such that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$ for all $1 \leq i \leq n, 1 \leq j \leq r$ and for some $d_{ij} \geq 0$. Then J has a finite Gröbner Σ-basis.*

Proof. Denote $I = \langle x_i(\sigma_j^{d_{ij}}) \mid 1 \leq i \leq n, 1 \leq j \leq r \rangle_\Sigma$ and $L = \mathrm{LM}(J)$. Then, we have that $I \subset L$ and the ideal $L/I \subset P/I$ has a finite basis owing to Proposition 6.7 and Proposition 6.9. In other words, the Σ-ideal L has a finite Σ-basis given by the finite Σ-basis of I together with the finite basis of L/I. $\qquad \square$

Note that the above result is generally not a necessary condition for finiteness of Gröbner Σ-bases. Consider for instance the example presented in Section 5 of [18]. Nevertheless, Corollary 6.10 guarantees termination of the algorithm SIGMAGBASIS when a complete set of variables $x_i(\sigma_j^{d_{ij}})$ for all i, j, occur as leading monomials of some elements of the Gröbner Σ-basis at some intermediate step of the computation. Of course, if such elements belongs to the input Σ-basis of a Σ-ideal $J \subset P$ one knows in advance that all properties of Noetherianity and termination are provided for the quotient $P' = P/J$. In particular, one may have that the polynomials $f_{ij} \in P$ such that $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$ are themselves a Gröbner Σ-basis of J. This is clearly the case when $J = \langle x_i(\sigma_j^{d_{ij}}) \mid 1 \leq i \leq n, 1 \leq j \leq r \rangle_\Sigma$. For all $d \geq 0$, define now

$$J^{(d)} = \langle x_i(\sigma) \mid 1 \leq i \leq n, \deg(\sigma) = d + 1 \rangle_\Sigma \supset \langle x_i(\sigma_j^{d+1}) \mid 1 \leq i \leq n, 1 \leq j \leq r \rangle_\Sigma$$

and put $J^{(-\infty)} = \langle X(1) \rangle_\Sigma = \langle X(\Sigma) \rangle$. If $P = \bigoplus_{d \in \hat{\mathbb{N}}} P_d$ is the grading of P defined by the order function then the subalgebra $P^{(d)} = \bigoplus_{i \leq d} P_i \subset P$ is clearly isomorphic to the quotient P/J_d and hence it can be endowed with the structure of Σ-algebra. In other words, to make use of the Noetherian (Σ-)subalgebras filtration

$$K = P^{(-\infty)} \subset P^{(0)} \subset P^{(1)} \subset \ldots \subset P$$

to perform concrete computations with Gröbner Σ-bases as explained in Section 4 corresponds to work progressively modulo the Σ-ideals

$$\langle X(\Sigma) \rangle = J^{(-\infty)} \supset J^{(0)} \supset J^{(1)} \supset \ldots \supset 0$$

providing a finite number of normal variables.

Another relevant case is the ordinary one that is when $\Sigma = \langle \sigma \rangle$. In this case, any set of polynomials $f_1, \ldots, f_n \in P$ such that $\mathrm{lm}(f_i) = x_i(\sigma^{d_i})$ $(d_i \geq 0)$ is a Gröbner Σ-basis since no S-polynomials are defined (up to product criterion). One has finally the following interesting case.

Proposition 6.11. *Assume that K is a field of constants and consider the linear polynomials $f_{ij} = \sum_{0 \leq k \leq d_{ij}} c_{ijk} x_i(\sigma_j^k) \in P$ where $c_{ijk} \in K$ and $c_{ijd_{ij}} = 1$, for all $1 \leq i \leq n, 1 \leq j \leq r$. Then $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$ and the set $\{f_{ij}\}$ is a Gröbner Σ-basis.*

Proof. Since $X(\Sigma)$ is endowed with a Σ-ordering, one has that $x_i(\sigma_j^k) \prec x_i(\sigma_j^l)$ if $k < l$ and hence $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$. Then, the only S-polynomials to be considered are

$$s = \mathrm{spoly}(\sigma_q^{d_{iq}} \cdot f_{ip}, \sigma_p^{d_{ip}} \cdot f_{iq}) = \sum_{0 \leq k < d_{ip}} c_{ipk} x_i(\sigma_q^{d_{iq}} \sigma_p^k) - \sum_{0 \leq l < d_{iq}} c_{iql} x_i(\sigma_p^{d_{ip}} \sigma_q^l),$$

for all $1 \leq i \leq n$ and $1 \leq p < q \leq r$. By reducing s with polynomials $\sigma_p^k \cdot f_{iq}$ and $\sigma_q^l \cdot f_{ip}$ one obtains

$$s' = - \sum_{0 \leq k < d_{ip}, 0 \leq l < d_{iq}} c_{ipk} c_{iql} x_i(\sigma_q^l \sigma_p^k) + \sum_{0 \leq l < d_{iq}, 0 \leq k < d_{ip}} c_{iql} c_{ipk} x_i(\sigma_p^k \sigma_q^l) = 0.$$

$\square$

Note explicitely that if Σ acts on K in a non-trivial way then generally

$$s' = - \sum_{0 \leq k < d_{ip}, 0 \leq l < d_{iq}} (\sigma_q^{d_{iq}} \cdot c_{ipk})(\sigma_p^k \cdot c_{iql}) x_i(\sigma_q^l \sigma_p^k) +$$
$$\sum_{0 \leq l < d_{iq}, 0 \leq k < d_{ip}} (\sigma_p^{d_{ip}} \cdot c_{iql})(\sigma_q^l \cdot c_{ipk}) x_i(\sigma_p^k \sigma_q^l) \neq 0.$$

7. A NOETHERIAN Σ-ALGEBRA OF SPECIAL INTEREST

From now on we assume that K is a field of constants. We define the ideal $J = \langle f_{ij} \rangle_\Sigma \subset P$ where $f_{ij} = \sum_{0 \leq k \leq d_{ij}} c_{ijk} x_i(\sigma_j^k)$ $(c_{ijk} \in K, c_{ijd_{ij}} = 1)$, for any $1 \leq i \leq n, 1 \leq j \leq r$. We want to describe the (Noetherian) Σ-algebra $P' = P/J$. To simplify notations and since they are interesting in itself, we consider separately the cases when $r = 1$ and $n = 1$. Then, assume $\Sigma = \langle \sigma \rangle$ and hence $P' = P/J$ where $J = \langle f_1, \ldots, f_n \rangle_\Sigma$ with $f_i = \sum_{0 \leq k \leq d_i} c_{ik} x_i(\sigma^k)$ $(c_{ik} \in K, c_{id_i} = 1)$. Define $Q = K[\sigma]$ the algebra of polynomials in the single variable σ and denote $g_i = \sum_{0 \leq k \leq d_i} c_{ik} \sigma^k \in Q$. Moreover, put $d = \sum_i d_i$ and let $V = K^d$. Finally, consider the $d \times d$ block-diagonal matrix

$$A = A_1 \oplus \ldots \oplus A_n = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

where each block A_i is the companion matrix of the polynomial g_i that is

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_{i0} \\ 1 & 0 & \dots & 0 & -c_{i1} \\ 0 & 1 & \dots & 0 & -c_{i2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{id-1} \end{pmatrix}.$$

Note that A has all entries in K and it can be considered as the Frobenius normal form of *any* $d \times d$ matrix provided that $g_1 \mid \dots \mid g_n$. Then, the monoid Σ or equivalently the algebra Q acts linearly over the vector space V by means of the representation $\sigma^k \mapsto A^k$. If $\{v_q\}_{1 \leq q \leq d}$ is the canonical basis of V, we denote $x_i(\sigma^k) = v_q$ where $q = \sum_{j < i} d_j + k + 1$ for all $1 \leq i \leq n, 0 \leq k < d_i$. We have hence $x_i(\sigma^k) = A^k x_i(1) = \sigma^k \cdot x_i(1)$. In other words, for the Q-module V one has the decomposition $V = \bigoplus_i V_i$ where V_i is the cyclic submodule generated by $x_i(1)$ and annihilated by the ideal $\langle g_i \rangle \subset Q$. Denote now by R' the (Noetherian) polynomial algebra generated by the finite set of variables $X(\Sigma)' = \{x_i(\sigma^k) \mid 1 \leq i \leq n, 0 \leq k < d_i\}$ that is V coincides with the subspace of linear forms of R'. Then, one extends the action of the monoid $\Sigma = \langle \sigma \rangle$ to the polynomial algebra R' in the natural way that is by putting, for all $k \geq 0$ and $x_i(\sigma^j) \in X(\Sigma)'$

$$\sigma^k \cdot x_i(\sigma^j) = A^k x_i(\sigma^j).$$

Denote by $\mathrm{end}(P)$ the algebra of all K-linear maps $P \to P$. Note that the representation $\rho : \Sigma \to \mathrm{End}(P)$ can be extended linearly to $\bar{\rho} : Q \to \mathrm{end}(P)$. Then, one has that $f_i = \sum_k c_{ik} x_i(\sigma^k) = \sum_k c_{ik} \sigma^k \cdot x_i(1) = g_i \cdot x_i(1)$, for all $i = 1, 2, \dots, n$.

Proposition 7.1. *If $\Sigma = \langle \sigma \rangle$ then the Σ-algebras P', R' are Σ-isomorphic.*

Proof. By Proposition 6.11 we have that the set $\{f_i\}$ is a Gröbner Σ-basis of the Σ-ideal $J \subset P$ and it is clear that the set of normal variables modulo J is exactly $X(\Sigma)' = \{x_i(\sigma^k) \mid 1 \leq i \leq n, 0 \leq k < d_i\}$. Moreover, since $R' \subset P$ and $f_i = g_i \cdot x_i(1)$ one has that $\mathrm{NF}(x_i(\sigma^k)) = \mathrm{NF}(\sigma^k \cdot x_i(1)) = A^k x_i(1)$, for all $k \geq 0$ and $x_i(1) \in X(1)'$. $\square$

Note that R' is Σ-generated by the set $X(1)' = \{x_i(1) \mid 1 \leq i \leq n, d_i > 0\}$. Since P is a free Σ-algebra, a surjective Σ-algebra homomorphism $\varphi : P \to R'$ is defined such that

$$x_i(1) \mapsto \begin{cases} x_i(1) & \text{if } d_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the above result states that the Σ-ideal $\mathrm{Ker}\,\varphi \subset P$ of all Σ-algebra relations satisfied by the generating set $X(1)' \cup \{0\}$ of R' is exactly J.

Let now $\Sigma = \langle \sigma_1, \dots, \sigma_r \rangle$ and suppose $X = \{x\}$. Then $P' = P/J$ where $J = \langle f_1, \dots, f_r \rangle_\Sigma$ with $f_j = \sum_{0 \leq k \leq d_j} c_{jk} x(\sigma_j^k)$ $(c_{jk} \in K, c_{jd_j} = 1)$. Define $Q = K[\sigma_1, \dots, \sigma_r]$ the algebra of polynomials in the variables σ_j and denote $g_j = \sum_{0 \leq k \leq d_j} c_{jk} \sigma_j^k \in Q$. One has clearly that $f_j = g_j \cdot x(1)$. As before, we consider the companion matrix A_j of the polynomial g_j in the single variable σ_j. If $d = \prod_j d_j$ then the monoid $\Sigma = \Sigma_1 \times \dots \times \Sigma_r$ $(\Sigma_j = \langle \sigma_j \rangle)$ that is the algebra $Q = Q_1 \otimes \dots \otimes Q_r$ $(Q_j = K[\sigma_j])$ acts linearly over the space $V = K^d$ my means of the representation

$$\sigma_1^{k_1} \cdots \sigma_r^{k_r} \mapsto A_1^{k_1} \otimes \dots \otimes A_r^{k_r},$$

where $A_1^{k_1} \otimes \cdots \otimes A_r^{k_r}$ denotes the Kronecker product of the matrices $A_j^{k_j}$. In other words, the Q-module V is the tensor product $V = V_1 \otimes \cdots \otimes V_r$ where V_j is the cyclic Q_j-module defined by the representation $\sigma_j^k \mapsto A_j^k$. If $\{v_{k_1} \otimes \cdots \otimes v_{k_r}\}_{1 \le k_j \le d_j}$ is the canonical basis of V, we put $x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) = v_{k_1+1} \otimes \cdots \otimes v_{k_r+1}$, for all $1 \le j \le r, 0 \le k_j < d_j$. One has then

$$x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) = (A_1^{k_1} \otimes \ldots \otimes A_r^{k_r})x(1) = (\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(1),$$

that is V is a cyclic module generated by $x(1)$. Denote now by R' the polynomial algebra generated by the finite set of variables $X(\Sigma)' = \{x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \mid 1 \le j \le r, 0 \le k_j < d_j\}$ that is V is the subspace of linear forms of R'. Again, we extend the action of the monoid $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ to the polynomial algebra R' by putting, for all $k_1, \ldots, k_r \ge 0$ and $x(\sigma) \in X(\Sigma)'$

$$(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(\sigma) = (A_1^{k_1} \otimes \cdots \otimes A_r^{k_r})x(\sigma).$$

Proposition 7.2. *If $X = \{x\}$ then P', R' are Σ-isomorphic.*

Proof. Assume $d \ne 0$ that is $d_j \ne 0$ for all j. Again, by Proposition 6.11 one has that the set $\{f_j\}$ is a Gröbner Σ-basis of $J \subset P$ and the set of normal variables modulo J is clearly $X(\Sigma)' = \{x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \mid 1 \le j \le r, 0 \le k_j < d_j\}$. Moreover, because $R' \subset P$ and $f_j = g_j \cdot x(1)$ we obtain that, for all $k_1, \ldots, k_r \ge 0$

$$\mathrm{NF}(x(\sigma_1^{k_1} \cdots \sigma_r^{k_r})) = \mathrm{NF}((\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(1)) = (A_1^{k_1} \otimes \ldots \otimes A_r^{k_r})x(1).$$

Finally, if $d = 0$ then $P' = R' = K$. $\qquad\square$

Note that for $d \ne 0$ one has that R' is Σ-generated by the element $x(1)$. Then, the above result implies that the Σ-ideal $J \subset P$ coincides with the ideal of Σ-algebra relations satisfied by the generator $x(1)$ that is the kernel of the Σ-algebra epimorphism $P \to R'$ such that $x(1) \mapsto x(1)$.

Consider finally the general case for the Σ-algebra $P' = P/J$ where $J = \langle f_{ij} \rangle_\Sigma$ and $f_{ij} = \sum_{0 \le k \le d_{ij}} c_{ijk} x_i(\sigma_j^k)$ with $c_{ijk} \in K, c_{ijd_{ij}} = 1$, for all $1 \le i \le n$ and $1 \le j \le r$. By combining the previous results, it is clear that such structure arises from the Q-module $V = K^d$ where $d = \sum_{1 \le i \le n} \prod_{1 \le j \le r} d_{ij}$ and the representation is given by the mapping

$$\prod_j \sigma_j^{k_j} \mapsto \bigoplus_i \bigotimes_j A_{ij}^{k_j}$$

where A_{ij} is the companion matrix of the polynomial $g_{ij} = \sum_{0 \le k \le d_{ij}} c_{ijk} \sigma_j^k$. In other words, we have that $V = \bigoplus_i \bigotimes_j V_{ij}$ where V_{ij} is the cyclic Q_j-module annihilated by the ideal $\langle g_{ij} \rangle \subset Q_j$. By denoting $x_i(1)$ the generator of the Q-module $\bigotimes_j V_{ij}$, we obtain that P' is isomorphic to the Σ-algebra $R' = K[X(\Sigma)']$ where $X(\Sigma)' = \{x_i(\sigma_1^{k_{i1}} \cdots \sigma_r^{k_{ir}}) \mid 1 \le i \le n, 1 \le j \le r, 0 \le k_{ij} < d_{ij}\}$ is the canonical basis of the space V. Then, one has that $J = \langle f_{ij} \rangle_\Sigma$ is exactly the Σ-ideal of Σ-algebra relations satisfied by generating set $X(1)' \cup \{0\}$ of R'.

8. ANOTHER EXAMPLE

With the aim of showing some application of the previous results, we fix now a setting that has been already considered in [23]. Note that in our approach all computations can be performed over any field. Fix $r = 1$ that is $\Sigma = \langle \sigma \rangle$ and $Q = K[\sigma]$. Consider $\mathbb{S}_d$ the symmetric group on d elements and let $\gamma \in \mathbb{S}_d$ be any

permutation. Denote $\Gamma = \langle \gamma \rangle \subset \mathbb{S}_d$ the cyclic subgroup generated by γ. Moreover, let $\gamma = \gamma_1 \cdots \gamma_n$ be the cycle decomposition of γ and denote d_i the length of the cycle γ_i. Consider the polynomial algebra $R' = K[x_i(\sigma^j) \mid 1 \leq i \leq n, 0 \leq j < d_i]$ and identify the subset $\{x_i(1), \ldots, x_i(\sigma^{d_i-1})\}$ with the support of the cycle γ_i. Clearly R' is a Γ-algebra that is there is a (faithful) group representation $\rho' : \Gamma \to \operatorname{Aut}(R')$. Consider now the polynomials $g_i = \sigma^{d_i} - 1 \in Q$ and define the $d \times d$ block-diagonal matrix

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

where each block A_i is the companion matrix of the polynomial g_i that is the permutation matrix

$$A_i = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

If we order the variables of R' as $x_1(1), \ldots, x_1(\sigma^{d_1-1}), \ldots, x_n(1), \ldots, x_n(\sigma^{d_n-1})$ then the representation ρ' is defined as $\gamma^k \cdot x_i(\sigma^j) = A^k x_i(\sigma^j)$, for all i, j, k. In other words, one has that R' is a Σ-algebra isomorphic to $P' = P/J$ where $J = \langle f_1, \ldots, f_n \rangle_\Sigma$ and $f_i = g_i \cdot x_i(1) = x_i(\sigma^{d_i}) - x_i(1) \in P$. Consider now a Γ-ideal (equivalently a Σ-ideal) $L' = \langle h_1, \ldots, h_m \rangle_\Gamma \subset R'$ and define the Σ-ideal $L = \langle h_1, \ldots, h_m, f_1, \ldots, f_n \rangle_\Sigma \subset P$. According with Definition 6.5 and the identification of R' with the quotient P' one has that $G' \subset L'$ is a Gröbner Γ-basis (equivalently Σ-basis) of L' if by definition $G' \cup \{f_1, \ldots, f_n\}$ is a Gröbner Σ-basis of L. In practice, the computation of G' is obtained by the algorithm SIGMAGBASIS that generalizes the Buchberger's procedure to the free Σ-algebra P. Note that such computation terminates owing to Corollary 6.10.

   To illustrate the method we fix now $\gamma = (12345678) \in \mathbb{S}_8$ and $K = \mathbb{Q}$. To simplify the variables notation we identify Σ with $\mathbb{N}$ that is $R' = K[x(0), x(1), \ldots, x(7)]$. Consider the following Γ-ideal of R'

$$L' = \langle x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2) \rangle_\Gamma =$$
$$\langle x(0)x(2) - x(1)^2, x(1)x(3) - x(2)^2, x(2)x(4) - x(3)^2, x(3)x(5) - x(4)^2,$$
$$x(4)x(6) - x(5)^2, x(5)x(7) - x(6)^2, x(7)^2 - x(0)x(6), x(1)x(7) - x(0)^2,$$
$$x(0)x(3) - x(1)x(2), x(1)x(4) - x(2)x(3), x(2)x(5) - x(3)x(4),$$
$$x(3)x(6) - x(4)x(5), x(4)x(7) - x(5)x(6), x(6)x(7) - x(0)x(5),$$
$$x(0)x(7) - x(1)x(6), x(2)x(7) - x(0)x(1) \rangle.$$

Note that $x(0)x(2) - x(1)^2, x(1)x(3) - x(2)^2, x(0)x(3) - x(1)x(2)$ are well-know equations of the twisted cubic in $\mathbb{P}^3$. Define now $f = x(8) - x(0) \in P$ and hence $R' = P' = P/J$ where $J = \langle f \rangle_\Sigma$. Then, a Gröbner Γ-basis (or Σ-basis) of L' is obtained by computing a Gröbner Σ-basis of the ideal

$$L = \langle x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2), f \rangle_\Sigma \subset P.$$

Fix for instance the lexicographic monomial ordering on P (hence on R') with $x(0) \prec x(1) \prec \ldots$ which is clearly a Σ-ordering. The usual minimal Gröbner basis

of L' consists of 54 elements whose leading monomials are

$$x(7)^2, x(6)x(7),$$
$$x(0)x(2) \to x(1)x(3) \to x(2)x(4) \to x(3)x(5) \to x(4)x(6) \to x(5)x(7),$$
$$x(0)x(3) \to x(1)x(4) \to x(2)x(5) \to x(3)x(6) \to x(4)x(7), x(2)x(7),$$
$$x(1)x(7), x(0)x(7), x(6)^3, x(0)x(4)^2 \to x(1)x(5)^2 \to x(2)x(6)^2,$$
$$x(0)^2x(4) \to x(1)^2x(5) \to x(2)^2x(6) \to x(3)^2x(7), x(0)^2x(6), x(0)x(6)^2,$$
$$x(1)x(6)^2, x(1)^2x(6), x(3)^2x(4) \to x(4)^2x(5) \to x(5)^2x(6),$$
$$x(4)x(5)^2 \to x(5)x(6)^2, x(0)x(1)x(6), x(0)x(4)x(5) \to x(1)x(5)x(6),$$
$$x(0)x(5)x(6), x(1)x(2)x(6), x(2)^4 \to x(3)^4 \to x(4)^4 \to x(5)^4, x(0)^3x(5),$$
$$x(0)x(5)^3, x(2)^3x(3), x(2)x(3)^3 \to x(3)x(4)^3, x(0)^2x(5)^2, x(0)^2x(1)x(5),$$
$$x(2)^2x(3)^2, x(1)^2x(2)^3, x(1)^4x(2)^2, x(1)^6x(2), x(1)^8.$$

Note that the arrow between two monomials means that a monomial can be obtained by the previous one by means of the Σ-action. Then, the minimal Gröbner Γ-basis of L' has just 32 elements and their leading monomials are

$$x(7)^2, x(6)x(7), x(0)x(2), x(0)x(3), x(2)x(7), x(1)x(7), x(0)x(7), x(6)^3,$$
$$x(0)x(4)^2, x(0)^2x(4), x(0)^2x(6), x(0)x(6)^2, x(1)x(6)^2, x(1)^2x(6), x(3)^2x(4),$$
$$x(4)x(5)^2, x(0)x(1)x(6), x(0)x(4)x(5), x(0)x(5)x(6), x(1)x(2)x(6), x(2)^4,$$
$$x(0)^3x(5), x(0)x(5)^3, x(2)^3x(3)x(2)x(3)^3, x(0)^2x(5)^2, x(0)^2x(1)x(5),$$
$$x(2)^2x(3)^2, x(1)^2x(2)^3, x(1)^4x(2)^2, x(1)^6x(2), x(1)^8.$$

Precisely, such basis is given by following elements

$$x(7)^2 - x(0)x(6), x(6)x(7) - x(0)x(5), x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2),$$
$$x(2)x(7) - x(0)x(1), x(1)x(7) - x(0)^2, x(0)x(7) - x(1)x(6), x(6)^3 - x(0)x(5)^2,$$
$$x(0)x(4)^2 - x(2)x(3)^2, x(0)^2x(4) - x(1)^2x(2), x(0)^2x(6) - x(0)x(1)x(5),$$
$$x(0)x(6)^2 - x(1)x(5)x(6), x(1)x(6)^2 - x(0)^2x(5), x(1)^2x(6) - x(0)^3,$$
$$x(3)^2x(4) - x(0)x(1)^2, x(4)x(5)^2 - x(0)x(1)x(5), x(0)x(1)x(6) - x(2)^2x(3),$$
$$x(0)x(4)x(5) - x(0)^2x(1), x(0)x(5)x(6) - x(3)x(4)^2,$$
$$x(1)x(2)x(6) - x(0)x(4)x(5), x(2)^4 - x(0)^4, x(0)^3x(5) - x(3)^3x(4),$$
$$x(0)x(5)^3 - x(3)x(4)^3, x(2)^3x(3) - x(0)^3x(1), x(2)x(3)^3 - x(0)x(1)^3,$$
$$x(0)^2x(5)^2 - x(2)^2x(3)^2, x(0)^2x(1)x(5) - x(3)^2x(4)^2, x(2)^2x(3)^2 - x(0)^2x(1)^2,$$
$$x(1)^2x(2)^3 - x(0)^5, x(1)^4x(2)^2 - x(0)^6, x(1)^6x(2) - x(0)^7, x(1)^8 - x(0)^8.$$

This computation can be performed by applying the algorithm SigmaGBasis to the Σ-ideal $L \subset P$ in the same way as for the example in Section 5. For details about different strategies to implement this method we refer to [18]. Note that Γ-ideals are also called "symmetric ideals" in [23].

9. Conclusions and further directions

In this paper we show that a viable theory of Gröbner bases exists for the algebra of partial difference polynomials which implies that one has symbolic (formal) computation for systems of partial difference equations. In fact, we prove that such Gröbner bases can be computed in a finite number of steps when truncated with respect to an appropriate grading or when they contain elements with suitable linear leading monomials. Precisely, since the algebras of difference polynomials are free objects in the category of Σ-algebras where Σ is a monoid isomorphic to $\mathbb{N}^r$, we obtain the latter result as a Noetherianity criterion for finitely generated Σ-algebras. Among such Noetherian Σ-algebras one finds polynomial algebras in a finite number of variables where a tensor product of a finite number of algebras generated

by single matrices acts over the subspace of linear forms. Considering that such commutative tensor algebras include group algebras of finite abelian groups one obtains that there exists a consistent Gröbner basis theory for ideals of finitely generated polynomial algebras that are invariant under such groups. In our opinion, this represents an interesting step in the direction of development of computational methods for ideals or algebras that are subject to group or algebra symmetries.

As for further developments, we may suggest that the study of important structures related to Gröbner bases like Hilbert series and free resolutions should be developed in the perspective that their definition and computation has to be consistent to the symmetry one finds eventually on a polynomial algebra. An important work in this direction is contained in [15]. Finally, the problem of studying conditions providing Σ-Noetherianity (instead of simple Noetherianity) for finitely generated Σ-algebras is also an intriguing subject.

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